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LETTER TO THE EDITOR

Non-Lie symmetry groups of (2+1)-dimensional nonlinear systems obtained from a simple direct method

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Abstract
A new direct method is developed to find finite symmetry groups of nonlinear mathematical physics systems. Using the direct method for the well-known (2+1)-dimensional Kadomtsev–Petviashvili equation and the Ablowitz–Kaup–Newell–Segur system, both the Lie point symmetry groups and the non-Lie symmetry groups are obtained. The Lie symmetry groups obtained via traditional Lie approaches are only special cases. Furthermore, the expressions of the exact finite transformations of the Lie groups are much simpler than those obtained via the standard approaches.

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1. Introduction

The study of symmetry is one of the most powerful methods in every branch of natural science especially in integrable systems because of the existence of infinitely many symmetries. To find the Lie point symmetry group of a nonlinear system, there is a standard method thanks to the famous first fundamental theorem of Lie [1]. And the standard method had been widely used to find Lie point symmetry algebras and groups for almost all the known integrable systems. However, to find non-Lie point symmetry groups is still quite difficult and there is little literature on the topic. Furthermore, even for the Lie point symmetry groups, the final known expressions may be quite complicated and difficult for real applications especially for physicists and other non-mathematical scientists.

In [2], Clarkson and Kruskal (CK) introduced a simple direct method to derive symmetry reductions of a nonlinear system without using any group theory. For many types of nonlinear systems, the method can be used to find *all* the possible similarity reductions. This fact hints that there is a simple method to find generalized symmetry groups for many types of nonlinear equations.

In this short letter, we modify CK's direct method to find the generalized Lie and non-Lie symmetry groups for the well-known Kadomtsev–Petviashvili (KP) equation and the Ablowitz–Kaup–Newell–Segur (AKNS) system which is a slightly general form of the Davey–Stewartson equation.

The main idea of [2] is to seek a reduction of a given partial differential equation (PDE) in the form

$$u(x_1, x_2, \ldots, x_n) = W(x_1, x_2, \ldots, x_n, U(z_1, z_2, \ldots, z_m)),$$
$$(z_i = z_i(x_1, x_2, \ldots, x_n), i = 1, 2, \ldots, m < n) \quad (1)$$

which is the most general form for a similarity reduction [3]. For a given PDE

$$F\big(x_i, u, u_{x_i}, u_{x_i x_j}, \ldots, i, j = 1, 2, \ldots, n\big) = 0, \quad (2)$$

substituting (1) into (2) and demanding that the result be a lower dimensional partial (or ordinary) differential equation for $U(z_1, z_2, \ldots, z_m)$ imposes conditions on W, z_i and their derivatives that enable one to solve for W and z_i.

It is interesting that for many real physical systems, it is enough to seek the symmetry reductions in a simple form

$$u(x_1, x_2, \ldots, x_n) = \alpha(x_1, x_2, \ldots, x_n) + \beta(x_1, x_2, \ldots, x_n)U(z_1, z_2, \ldots, z_m) \quad (3)$$

instead of the general form (1).

CK's direct symmetry reduction method implies that it is possible to find full symmetry groups of the KP equation and AKNS system by a simple direct method without using any group theory.

To realize this idea is also quite straightforward. In fact, the only thing one has to do is to substitute

$$u(x_1, x_2, \ldots, x_n) = W(x_1, x_2, \ldots, x_n, U(X_1, X_2, \ldots, X_n)),$$
$$(X_i = X_i(x_1, x_2, \ldots, x_n), i = 1, 2, \ldots, n) \quad (4)$$

into a given PDE (2) and require U satisfies the same PDE under the transformation $\{u, x_1, \ldots, x_n\} \to \{U, X_1, \ldots, X_n\}$.

It is not surprising but interesting that it is enough to suppose that the group transformation has some simple forms, say,

$$u(x_1, x_2, \ldots, x_n) = \alpha(x_1, x_2, \ldots, x_n) + \beta(x_1, x_2, \ldots, x_n)U(X_1, X_2, \ldots, X_n) \quad (5)$$

instead of (4) for various nonlinear systems especially for the models that CK's direct reduction method does work.

In section 2, we show the simple method with a proof of the generality of ansatz (5) for the KP equation. In section 3, the method is further used for the AKNS system. The last section is a short summary and discussion.

2. Transformation group of the KP equation

To show the validity of the method, we first take the KP equation

$$u_{xt} + (u_{xxx} - 6uu_x)_x + 3u_{yy} = 0, \quad (6)$$

as a simple illustration model. The KP equation comes from the study of long gravity waves in a single layer, or multilayered shallow fluid, when the waves propagate predominantly in one direction with a small perturbation in the perpendicular one. It also appeared in many other fields, for instance, in plasma physics, gas dynamics and so on. It has been shown that the KP equation is one of the few integrable equations in high dimensions in the sense of allowing a Lax pair, an infinity of conservation laws, soliton and multisoliton solutions etc [8–12].

In [7], David, Levi and Winternitz studied the Lie point symmetry group of the KP equation via the traditional Lie group approach. The generalized w_∞ symmetry algebra is given in [13] and the symmetry reductions of the model are given in [14, 15].

First, we should prove that the assumption

$$u = \alpha + \beta U(\xi, \eta, \tau), \tag{7}$$

where $\alpha, \beta, \xi, \eta, \tau$ are functions of x, y, t is enough instead of the most general one,

$$u = W(x, y, t, U(\xi, \eta, \tau)), \tag{8}$$

for the KP equation.

Substituting (8) into (6) and requiring $U(\xi, \eta, \tau)$ also is a solution of the KP equation but with independent variables (eliminating $U_{\xi\tau}$ and its higher order derivatives by means of the KP equation), we have

$$-4W_U\xi_x\tau_x^3U_{\xi^{10}} + G(x, y, t, U, U_\xi, \ldots, U_{\xi^9}, U_\eta, U_\tau, U_{\tau\tau}, \ldots) = 0, \tag{9}$$

where $U_{\xi^n} \equiv \frac{\partial^n U}{\partial \xi^n}$, G is a complicated $U_{\xi^{10}}$ (and $U_{\xi\tau}$ and its higher order derivatives) independent function. Equation (9) is true for an arbitrary solution U only for all coefficients of the polynomials of the derivatives of U being zero.

Obviously, W_U should not be zero and one can easily prove that there is no nontrivial solution for $\xi_x = 0$, so causing the coefficients of $U_{\xi^{10}}$ to vanish, the only possible case is

$$\tau_x = 0, \qquad \text{i.e.} \quad \tau \equiv \tau(y, t). \tag{10}$$

Using condition (10), (9) is reduced to

$$W_U\left(3\tau_y^2U_{\tau\tau} + \eta_x^4U_{\eta^4}\right) + G_1(x, y, t, U, U_\xi, \ldots, U_{\xi^4}, U_\eta, U_\tau, \ldots) = 0, \tag{11}$$

where G_1 is a $U_{\tau\tau}$ and U_{η^4} independent function. Vanishing the coefficients of $U_{\tau\tau}$ and U_{η^4} yields

$$\tau \equiv \tau(t), \qquad \eta \equiv \eta(y, t). \tag{12}$$

Under condition (12), (11) is further simplified to

$$W_{UU}\xi_x^4U_{\xi\xi}^2 + G_2(x, y, t, U, U_\xi)U_{\xi\xi} + G_3(x, y, t, U, U_\xi, U_{\xi^3}, U_{\xi^4}, U_\eta, U_{\eta\eta}, U_\tau, U_{\xi\eta}) = 0, \tag{13}$$

where G_2 and G_3 are only functions of the indicated functions. Now vanishing the coefficient of $U_{\xi\xi}^2$, i.e., W_{UU}, proves the conclusion that assumption (7) instead of the general one (8) is sufficient to find the general symmetry group of the KP equation.

Now the substitution of (7) with (12) into the KP equation leads to

$$\begin{aligned}
\beta\xi_x\left(\xi_x^3 - \tau_t\right)U_{\xi\xi\xi\xi} &+ (6\beta_y\eta_y + 3\beta\eta_{yy} + \beta_x\eta_t)U_\eta - 6(\beta\beta_x)_xU^2 + [6\beta\xi_x(\tau_t - \beta\xi_x)U_{\xi\xi} \\
&- 6\beta(\xi_{xx}\beta + 4\xi_x\beta_x)U_\xi - 6\alpha\beta_{xx} + \beta_{xxxx} - 12\alpha_x\beta_x + 3\beta_{yy} + \beta_{tx} - 6\beta\alpha_{xx}]U \\
&- 6\beta\xi_x(\beta\xi_x - \tau_t)U_\xi^2 + [6\beta_y\xi_y + \beta\xi_{tx} - 12\alpha_x\beta\xi_x - 12\beta_x\xi_x\alpha + \beta_x\xi_t + 6\xi_{xx}\beta_{xx} \\
&- 6\xi_{xx}\beta\alpha + 3\beta\xi_{yy} + \beta\xi_{xxxx} + \beta_t\xi_x + 4\beta_{xxx}\xi_x + 4\beta_x\xi_{xxx}]U_\xi + \beta_xU_\tau\tau_t \\
&+ \left[6\left(\xi_x^2\beta_x\right)_x + \beta\xi_x\xi_t + 3\beta\xi_{xx}^2 - 6\beta\xi_x^2\alpha + 4\beta\xi_x\xi_{xxx} + 3\beta\xi_y^2\right]U_{\xi\xi} \\
&+ 2\xi_x^2(2\xi_x\beta_x + 3\xi_{xx}\beta)U_{\xi\xi\xi} + \beta(6\xi_y\eta_y + \eta_t\xi_x)U_{\xi\eta} \\
&+ 3\beta U_{\eta\eta}\left(\eta_y^2 - \xi_x\tau_t\right) + (\alpha_t + \alpha_{xxx} - 6\alpha_x\alpha)_x + 3\alpha_{yy} = 0.
\end{aligned} \tag{14}$$

From (14), the remained determining equations of the functions $\xi, \eta, \tau, \alpha, \beta$ can be read off by vanishing the coefficients of the polynomials of U and its derivatives:

$$\xi_x^3 - \tau_t = 0, \qquad 6\beta_y\eta_y + 3\beta\eta_{yy} + \beta_x\eta_t = 0, \qquad (\beta\beta_x)_x = 0, \qquad \tau_t - \beta\xi_x = 0, \tag{15}$$

$$\xi_{xx}\beta + 4\xi_x\beta_x = 0, \qquad -6(\alpha\beta)_{xx} + \beta_{xxxx} + 3\beta_{yy} + \beta_{tx} = 0, \qquad \beta_x = 0, \tag{16}$$

$$6\beta_y\xi_y + \beta\xi_{tx} - 12\xi_x(\alpha\beta)_x + \beta_x\xi_t + 6\xi_{xx}(\beta_{xx} - \beta\alpha) + 3\beta\xi_{yy} \tag{17}$$

$$+ \beta\xi_{xxxx} + \beta_t\xi_x + 4\beta_{xxx}\xi_x + 4\beta_x\xi_{xxx} = 0, \tag{18}$$

$$6\left(\xi_x^2\beta_x\right)_x + \beta\left(\xi_x\xi_t + 3\xi_{xx}^2 - 6\xi_x^2\alpha + 4\xi_x\xi_{xxx} + 3\xi_y^2\right) = 0, \qquad \xi_x\beta_x + 3\xi_{xx}\beta = 0, \tag{19}$$

$$6\xi_y\eta_y + \eta_t\xi_x = 0, \qquad \eta_y^2 - \xi_x\tau_t = 0, \qquad (\alpha_t + \alpha_{xxx} - 6\alpha_x\alpha)_x + 3\alpha_{yy} = 0. \tag{20}$$

It is straightforward to find out the general solution of the determining equations (15)–(20). The result reads

$$\xi = C_1\tau_t^{1/3}x - \frac{1}{18}C_1\tau_t^{-2/3}\tau_{tt}y^2 - \frac{\delta\eta_{0t}y}{6C_1\tau_t^{1/3}} + \xi_0, \tag{21}$$

$$\eta = \delta C_1^2\tau_t^{2/3}y + \eta_0, \qquad \beta = C_1^2\tau_t^{2/3} \tag{22}$$

$$\alpha = \frac{1}{18}(\ln\tau_t)_t x - \frac{\left(3\tau_t\tau_{ttt} - 4\tau_{tt}^2\right)y^2}{324\tau_t^2} - \frac{C_1\delta y\tau_t^{1/3}}{36}\left(\frac{\eta_{0t}}{\tau_t}\right)_t + \frac{C_1^2\left(\eta_{0t}^2 + 12\xi_{0t}\tau_t\right)}{72\tau_t^{4/3}}, \tag{23}$$

where $\xi_0 \equiv \xi_0(t)$, $\eta_0 \equiv \eta_0(t)$ and $\tau \equiv \tau(t)$ are arbitrary functions of time t while the constants δ and C_1 possess discrete values determined by

$$\delta = \pm 1, \qquad C_1 = 1, \qquad -\tfrac{1}{2}(1 \pm \mathrm{i}\sqrt{3}), \qquad (\mathrm{i} = \sqrt{-1}). \tag{24}$$

In summary, the following theorem holds:

Theorem 1. *If $U = U(x, y, t)$ is a solution of the KP equation (6) then so is*

$$u = \frac{1}{18}(\ln\tau_t)_t x - \frac{\left(3\tau_t\tau_{ttt} - 4\tau_{tt}^2\right)y^2}{324\tau_t^2} - \frac{C_1\delta y\tau_t^{1/3}}{36}\left(\frac{\eta_{0t}}{\tau_t}\right)_t + \frac{C_1^2\left(\eta_{0t}^2 + 12\xi_{0t}\tau_t\right)}{72\tau_t^{4/3}}$$

$$+ C_1^2\tau_t^{2/3}U(\xi, \eta, \tau), \tag{25}$$

with (21) and (22), where ξ_0, η_0 and τ are arbitrary functions of t and the discrete value of the constants δ and C_1 are given by (24).

From the symmetry group theorem 1, we know that for the real KP equation, the symmetry group is divided into two sectors: the Lie point symmetry group which corresponds to

$$\delta = C_1 = 1 \tag{26}$$

and a coset of the Lie group which is related to

$$\delta = -1, \qquad C_1 = 1. \tag{27}$$

The coset is equivalent to the reflected transformation of y, i.e., $y \to -y$ company with the usual Lie point symmetry transformation.

In other words, if we denote by $\mathcal{S}$ the Lie point symmetry group of the real KP equation, by σ^y the reflection of y, by I the identity transformation and by $\mathcal{C}_2 \equiv \{I, \sigma^y\}$ the discrete reflection group, then the full Lie symmetry group $\mathcal{G}_{\mathrm{RKP}}$ of the real KP equation given by theorem 1 can be expressed as

$$\mathcal{G}_{\mathrm{RKP}} = \mathcal{C}_2 \otimes \mathcal{S}. \tag{28}$$

For the complex KP equation, the symmetry group is divided into six sectors which correspond to

$$\delta = 1, \qquad C_1 = 1, \tag{29}$$
$$\delta = 1, \qquad C_1 = \tfrac{1}{2}(\mathrm{i}\sqrt{3} - 1), \tag{30}$$
$$\delta = 1, \qquad C_1 = \tfrac{1}{2}(\mathrm{i}\sqrt{3} + 1), \tag{31}$$
$$\delta = -1, \qquad C_1 = 1, \tag{32}$$
$$\delta = -1, \qquad C_1 = \tfrac{1}{2}(\mathrm{i}\sqrt{3} - 1), \tag{33}$$

and

$$\delta = -1, \qquad C_1 = \tfrac{1}{2}(\mathrm{i}\sqrt{3} + 1), \tag{34}$$

of theorem 1 respectively. That is to say, the full symmetry group, $\mathcal{G}_{\mathrm{CKP}}$, expressed by theorem 1 for the complex KP equation is the product of the usual Lie point symmetry group $\mathcal{S}$ (theorem 1 with (29)) and the discrete group $\mathcal{D}_3$

$$\mathcal{G}_{\mathrm{CKP}} = \mathcal{D}_3 \otimes \mathcal{S}, \tag{35}$$
$$\mathcal{D}_3 \equiv \{I, \sigma^y, R_1, R_2, \sigma^y R_1, \sigma^y R_2\}, \tag{36}$$

where I is the identity transformation, σ^y is the reflection of y and

$$R_1: \quad u(x, y, t) \to -\tfrac{1}{2}(1 + \mathrm{i}\sqrt{3})u\big(\tfrac{1}{2}(\mathrm{i}\sqrt{3} - 1)x, -\tfrac{1}{2}(1 + \mathrm{i}\sqrt{3})y, t\big), \tag{37}$$
$$R_2: \quad u(x, y, t) \to \tfrac{1}{2}(\mathrm{i}\sqrt{3} - 1)u\big(-\tfrac{1}{2}(\mathrm{i}\sqrt{3} + 1)x, \tfrac{1}{2}(\mathrm{i}\sqrt{3} - 1)y, t\big). \tag{38}$$

For the special sector of the theorem, the Lie point symmetry group of the KP equation, an equivalent form but with a much more complicated expression had been given in the literature, say [4], by means of the standard Lie approach. To see the equivalence between the Lie point symmetry group obtained in theorem 1 with (29) and the known one from the traditional method, we set

$$\tau(t) = t + \epsilon f(t), \qquad \xi_0(t) = \epsilon g(t), \qquad \eta_0(t) = \epsilon h(t), \qquad \delta = 1, \qquad C_1 = 1, \tag{39}$$

with an infinitesimal parameter ϵ, then (25) can be written as

$$\begin{aligned} u &= U + \epsilon\sigma(U) \\ \sigma(U) &= \big(-\tfrac{1}{6}g_t(t)y + h(t) + \tfrac{1}{3}f_t(t)x - \tfrac{1}{18}f_{tt}(t)y^2\big)U_x + \big(\tfrac{2}{3}yf_t(t) + g(t)\big)U_y \\ &\quad + f(t)U_t + \tfrac{2}{3}f_t(t)U + \tfrac{1}{36}yg_{tt}(t) - \tfrac{1}{18}f_{tt}(t)x - \tfrac{1}{6}h_t(t) + \tfrac{1}{108}f_{ttt}(t)y^2. \end{aligned} \tag{40}$$

The equivalent vector expression of the above symmetry is

$$\begin{aligned} V = {} & \left\{\left(\frac{1}{3}f_t(t)x - \frac{1}{18}f_{tt}(t)y^2\right)\frac{\partial}{\partial x} + \frac{2}{3}yf_t(t)\frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial t}\right. \\ & \left. + \left(\frac{1}{18}f_{tt}(t)x - \frac{1}{108}f_{ttt}(t)y^2 - \frac{2}{3}f_t(t)U\right)\frac{\partial}{\partial U}\right\} \\ & + \left\{-\frac{1}{6}g_t(t)y\frac{\partial}{\partial x} + g(t)\frac{\partial}{\partial y} - \frac{1}{36}g_{tt}(t)y\frac{\partial}{\partial U}\right\} \\ & + \left\{h(t)\frac{\partial}{\partial x} - \frac{1}{6}g_t(t)\frac{\partial}{\partial U}\right\} \equiv V_1(f(t)) + V_2(g(t)) + V_3(h(t)), \end{aligned}$$

which is exactly the same as that obtained by the standard Lie approach.

3. Transformation group and symmetry algebra of the AKNS equation

The (2+1)-dimensional AKNS system

$$\mathrm{i}Q_t + \sigma_3(Q_{yy} - Q_{xx}) + [A, Q] = 0, \tag{41}$$

$$\begin{pmatrix} u_x & 0 \\ 0 & v_y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (\det Q)_y & 0 \\ 0 & (\det Q)_x \end{pmatrix}, \tag{42}$$

where

$$Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

is a natural (2+1)-dimensional generalization of the (1+1)-dimensional nonlinear Schrödinger (NLS) equation. The component form of (41) and (42) can be explicitly rewritten as

$$\begin{aligned} u_x &= -\tfrac{1}{2}(qr)_y, \qquad & \mathrm{i}r_t + r_{xx} - r_{yy} + r(v-u) = 0, \\ v_y &= -\tfrac{1}{2}(qr)_x, \qquad & \mathrm{i}q_t + q_{yy} - q_{xx} + q(u-v) = 0. \end{aligned} \tag{43}$$

When we restrict q as a complex conjugate of r, then the AKNS system is just the well-known Davey–Stewartson equation.

Similar to the KP equation, after finishing some quite tedious calculations, one can find that the simplified symmetry transformation ansatz has the form

$$\begin{aligned} r &= \beta_1 R(\xi,\eta,\tau) + \gamma_1 Q(\xi,\eta,\tau), & q &= \beta_2 Q(\xi,\eta,\tau) + \gamma_2 R(\xi,\eta,\tau), \\ u &= a_1 + \beta_3 U(\xi,\eta,\tau) + \gamma_3 V(\xi,\eta,\tau), & v &= a_2 + \beta_4 V(\xi,\eta,\tau) + \gamma_4 U(\xi,\eta,\tau), \end{aligned} \tag{44}$$

where $a_1, a_2, \beta_1, \beta_2, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \xi, \eta$ and τ are functions of $\{x, y, t\}$ only.

Substituting (44) into (43) and restricting $U(\xi,\eta,\tau) \equiv U$, $V(\xi,\eta,\tau) \equiv V$, $R(\xi,\eta,\tau) \equiv R$ and $Q(\xi,\eta,\tau) \equiv Q$ satisfies the same form as the AKNS system (43) but with new independent variables, i.e.,

$$\begin{aligned} R_{\xi\xi} &= R_{\eta\eta} - \mathrm{i}R_\tau + RU - VR, \\ Q_{\eta\eta} &= Q_{\xi\xi} - \mathrm{i}Q_\tau + QV - UQ, \\ U_\xi &= -\tfrac{1}{2}QR_\eta - \tfrac{1}{2}RQ_\eta, \\ V_\eta &= -\tfrac{1}{2}RQ_\xi - \tfrac{1}{2}QR_\xi, \end{aligned} \tag{45}$$

we can arrive at the following final transformation group theorem of the AKNS system (43):

Theorem 2. *If $\{U = U(x,y,t), V = V(x,y,t), Q = Q(x,y,t), R = R(x,y,t)\}$ is a solution of the AKNS system (43), so are $\{u_1, v_1, q_1, r_1\}$, $\{u_2, v_2, q_2, r_2\}$ and $\{u_3, v_3, q_3, r_3\}$ with*

$$r_1 = f_t b \exp\left(\frac{\mathrm{i}y\left[yf_{tt} + 4\delta_2\eta_{0t}f_t^{\frac{1}{2}}\right]}{8f_t} - \frac{\mathrm{i}x\left[xf_{tt} - 4\delta_1\xi_{0t}f_t^{\frac{1}{2}}\right]}{8f_t}\right) R(\xi,\eta,\tau)$$

$$q_1 = \frac{\delta_1\delta_2}{b}\exp\left(-\frac{\mathrm{i}y\left[yf_{tt} + 4\delta_2\eta_{0t}f_t^{\frac{1}{2}}\right]}{8f_t} + \frac{\mathrm{i}x\left[xf_{tt} + 4\delta_1\xi_{0t}f_t^{\frac{1}{2}}\right]}{8f_t}\right) Q(\xi,\eta,\tau)$$

$$\begin{aligned} u_1 = a_0 &+ \frac{f_{tt}\mathrm{i}}{f_t} + \frac{3}{16}\frac{y^2f_{tt}^2}{f_t^2} + \frac{1}{2}\frac{y\delta_2 f_{tt}\eta_{0t}}{f_t^{\frac{3}{2}}} + \frac{1}{4}\frac{\eta_{0t}^2}{f_t} - \frac{1}{4}\frac{\xi_{0t}^2}{f_t} - \frac{1}{2}\frac{\mathrm{i}f_{tt}}{f_t} \\ &- \frac{1}{8}\frac{y^2f_{ttt}}{f_t} - \frac{1}{2}\frac{y\delta_2\eta_{0tt}}{f_t^{\frac{1}{2}}} + f_tU(\xi,\eta,\tau), \end{aligned}$$

$$v_1 = a_0 + \frac{3}{16}\frac{x^2 f_{tt}^2}{f_t^2} - \frac{1}{8}\frac{x^2 f_{ttt}}{f_t} + \frac{1}{2}\frac{x\delta_1 f_{tt}\xi_{0t}}{f_t^{\frac{3}{2}}} - \frac{1}{2}\frac{x\delta_1\xi_{0tt}}{f_t^{\frac{1}{2}}} + f_t V(\xi,\eta,\tau),$$

$$\xi = \delta_1 f_t^{\frac{1}{2}} x + \xi_0, \qquad \eta = \delta_2 f_t^{\frac{1}{2}} y + \eta_0, \qquad \tau = f, \tag{46}$$

$$r_2 = \frac{\delta_1\delta_2 f_t}{b}\exp\left(\frac{-\mathrm{i} f_{tt}}{8 f_t}(x^2 - y^2) - \frac{\mathrm{i}}{2\sqrt{f_t}}(x\eta_{0t}\delta_1 - y\delta_2\xi_{0t})\right) R(X,Y,T) \tag{47}$$

$$q_2 = b\exp\left(\frac{\mathrm{i}}{8 f_t}(x^2 - y^2) f_{tt} + \frac{\mathrm{i}}{2\sqrt{f_t}}(-y\delta_2\xi_{0t} + x\eta_{0t}\delta_1)\right) Q(X,Y,T) \tag{48}$$

$$u_2 = a_0 - \frac{\sqrt{f_t}\,y}{8}\left(\frac{f_{tt}\,y}{f_t^{\frac{3}{2}}} + \frac{4\delta_2\xi_{0t}}{f_t}\right)_t + \frac{\mathrm{i} f_{tt}}{2 f_t} - \frac{\mathrm{i} b_t}{b} + \frac{\xi_{0t}^2 - \eta_{0t}^2}{4 f_t} + f_t V(X,Y,T), \tag{49}$$

$$v_2 = v = a_0 - \frac{\sqrt{f_t}\,x}{8}\left(\frac{f_{tt}}{f_t^{\frac{3}{2}}}x + 4\delta_1\frac{\eta_{0t}}{f_t}\right)_t + f_t U(X,Y,T), \tag{50}$$

$$X = \delta_2 f_t^{\frac{1}{2}} y + \xi_0, \qquad Y = \delta_1 f_t^{\frac{1}{2}} x + \eta_0, \qquad T = -f, \tag{51}$$

$$r_3 = b\exp\left(-\frac{\mathrm{i}}{2\sqrt{f_t}}(\delta_1\eta_{0t}x - \delta_2\xi_{0t}y) - \frac{\mathrm{i}(y^2 - x^2) f_{tt}}{8 f_t}\right) Q(x',y',t') \tag{52}$$

$$q_3 = \frac{\delta_1\delta_2 f_t}{b}\exp\left(\frac{\mathrm{i}}{2\sqrt{f_t}}(\delta_1\eta_{0t}x - \delta_2\xi_{0t}y) - \frac{\mathrm{i}(y^2 - x^2) f_{tt}}{8 f_t}\right) R(x',y',t') \tag{53}$$

$$u_3 = a_0 - \frac{\sqrt{f_t}\,y}{8}\left(\frac{f_{tt}\,y}{f_t^{\frac{3}{2}}} + \frac{4\delta_2\xi_{0t}}{f_t}\right)_t + \frac{\mathrm{i} b_t}{b} - \frac{\mathrm{i} f_{tt}}{2 f_t} + \frac{\xi_{0t}^2 - \eta_{0t}^2}{4 f_t} + f_t V(x',y',t'), \tag{54}$$

$$v_3 = a_0 - \frac{\sqrt{f_t}\,x}{8}\left(\frac{f_{tt}}{f_t^{\frac{3}{2}}}x + 4\delta_1\frac{\eta_{0t}}{f_t}\right)_t + f_t U(x',y',t'), \tag{55}$$

$$x' = \delta_2 f_t^{\frac{1}{2}} y + \xi_0, \qquad y' = \delta_1 f_t^{\frac{1}{2}} x + \eta_0, \qquad t' = f, \tag{56}$$

where $a_0 \equiv a_0(t)$, $f \equiv f(t)$, $\xi_0 \equiv \xi_0(t)$, $\eta_0 \equiv \eta_0(t)$ *and* $b \equiv b(t)$ *are all arbitrary functions of t and*

$$\delta_1^2 = 1, \qquad \delta_2^2 = 1. \tag{57}$$

From theorem 2, it is easy to see that the symmetry group, $\mathcal{G}_{\mathrm{AKNS}}$, of the AKNS system is divided into eight sectors which can be considered as a product of four discrete $\mathcal{C}_2$ groups and the usual Lie point symmetry group $\mathcal{S}$ which is related to transformations $\{r,q,u,v\} \to \{r_1,q_1,u_1,v_1\}$ with $\delta_1 = \delta_2 = 1$. That means

$$\mathcal{G}_{\mathrm{AKNS}} = \mathcal{C}_2^x \otimes \mathcal{C}_2^y \otimes \mathcal{C}_2^t \otimes \mathcal{C}_2^r \otimes \mathcal{S}, \tag{58}$$

where

$$\mathcal{C}_2^x \equiv \{I, \Sigma^x\}, \qquad \Sigma^x : \{x,q\} \to \{-x,-q\}, \tag{59}$$

$$\mathcal{C}_2^y \equiv \{I, \Sigma^y\}, \qquad \Sigma^y : \{y,q\} \to \{-y,-q\}, \tag{60}$$

$$\mathcal{C}_2^t \equiv \{I, \Sigma^t\}, \qquad \Sigma^t : \{x,y,t,u,v\} \to \{y,x,-t,v,u\}, \tag{61}$$

$$\mathcal{C}_2^r \equiv \{I, \Sigma^r\}, \qquad \Sigma^r : \{x, y, r, q, u, v\} \to \{y, x, q, r, v, u\}, \tag{62}$$

$$\mathcal{S} : \{r, q, u, v\} \to \{r_1, q_1, u_1, v_1\}|_{\delta_1=\delta_2=1}\,. \tag{63}$$

It should be mentioned that some other types of reflection transformations can be read off from (58) by taking the continuous functions as some special constants. For instance, transformations $\{x, r\} \to \{-x, -r\}$ and $\{y, r\} \to \{-y, -r\}$, can be read off from $\Sigma^x\mathcal{S}$ and $\Sigma^y\mathcal{S}$ respectively by taking $b = -1, f = \xi_0 = \eta_0 = a_0 = 0, \delta_1 = -1, \delta_2 = 1$ and $b = -1, f = \xi_0 = \eta_0 = a_0 = 0, \delta_1 = 1, \delta_2 = -1$.

A complicated but equivalent form of the Lie point symmetry sector, $\mathcal{S}$ with $b = 1$, for the DS system ($q = r^*$) had been given by some authors by using the standard Lie approach [4].

Whence the finite Lie point symmetry group is obtained, to find its related Lie symmetry algebra is quite straightforward. For the AKNS system, the Lie point symmetries have the forms

$$\sigma \equiv \sigma_1(f) + \sigma_2(g) + \sigma_3(h) + \sigma_4(n) + \sigma_5(m)$$
$$\equiv \begin{pmatrix} fu_t + \frac{1}{2}f_t(xu_x + yu_y + 2u) + \frac{\mathrm{i}}{2}f_{tt} - \frac{1}{8}y^2 f_{ttt} \\ fv_t + \frac{1}{2}f_t(xv_x + yv_y + 2v) - \frac{1}{8}x^2 f_{ttt} \\ fq_t + \frac{1}{2}f_t(xq_x + yq_y) + \frac{\mathrm{i}}{8}f_{tt}(x^2 - y^2)q \\ fr_t + \frac{1}{2}f_t(xr_x + yr_y + r) - \frac{\mathrm{i}}{8}f_{tt}(x^2 - y^2)r \end{pmatrix}$$
$$+ \begin{pmatrix} gu_x \\ gv_x - \frac{1}{2}xg_{tt} \\ gq_x + \frac{\mathrm{i}}{2}g_t xq \\ gr_x - \frac{\mathrm{i}}{2}g_t xr \end{pmatrix} + \begin{pmatrix} hu_y - \frac{1}{2}yh_{tt} \\ hv_y \\ hq_y - \frac{\mathrm{i}}{2}h_t yq \\ hr_y + \frac{\mathrm{i}}{2}h_t yr \end{pmatrix} + \begin{pmatrix} n \\ n \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{i}m_t \\ 0 \\ -mq \\ mr \end{pmatrix},$$

which can be obtained from theorem 2 by restricting ($f \equiv f(t), g \equiv g(t), h \equiv h(t), n \equiv n(t), m \equiv m(t)$)

$$\tau = t + \epsilon f, \qquad \xi_0 = \epsilon g, \qquad \eta_0 = \epsilon h, \qquad a_0 = \epsilon n, \qquad b_0 = 1 + \epsilon m.$$

The nonzero commutators among $\sigma_1(f), \sigma_2(g), \sigma_3(h), \sigma_4(n)$ and $\sigma_5(m)$ are

$$[\sigma_1(f_1), \sigma_1(f_2)] = \sigma_1(f_1 f_{2t} - f_{1t} f_2), [\sigma_1(f), \sigma_2(g)] = \sigma_2(f g_t),$$
$$[\sigma_1(f), \sigma_3(h)] = \sigma_3(f h_t), [\sigma_1(f), \sigma_4(n)] = \sigma_4(f n_t + f_t n),$$
$$[\sigma_1(f), \sigma_5(m)] = \sigma_5(f m_{tt} + f_t m_t).$$

Comparing the symmetries and the symmetry algebra of this paper under the condition $b = 1, m = 0$ with that of [4], one can easily find the equivalence between them.

4. Summary and discussions

In summary, both the Lie point symmetry group and the non-Lie symmetry group of many nonlinear systems can be obtained by some type of simple direct ansatz. If all symmetry reductions of a nonlinear system can be obtained by CK's direct method, then at least the full

Lie point symmetry group of the model can be obtained by the simple ansatz (5). Furthermore, after some concrete analysis we find that for the single component models, such as the KP equation, the KdV equation, the Boussinesq equation etc, ansatz (5) is also enough to find the general non-Lie symmetry groups, while for the multi-component nonlinear systems, some minor modifications are needed. For instance, for the (2+1)-dimensional AKNS system, the basic ansatz (5) has to be modified to the form (44) in order to find its generalized non-Lie symmetry group.

Though the generalized Lie point symmetry group of the nonlinear systems can be obtained from the standard Lie algebra theory, the final expressions obtained by means of the simple direct method proposed in this letter are much simpler than those obtained from the traditional approaches.

For the real valued KP equation, the symmetry group is a product of a discrete $\mathcal{C}_2$ group and an infinite dimensional Kac–Moody–Virasoro type Lie group with three arbitrary functions. However, for the complex valued KP equation, its full known Lie group in form (8) or its equivalent form (7) is a product of two discrete group ($\mathcal{C}_2$ and $\mathcal{C}_3$) and the infinite dimensional Kac–Moody–Virasoro type Lie group. For the (2+1)-dimensional AKNS system, the symmetry group is a product of four discrete $\mathcal{C}_2$ groups and one infinite dimensional Kac–Moody–Virasoro type Lie group with five arbitrary functions. A similar property is valid for all other known (2+1)-dimensional integrable models.

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